Lecture 5 01/31/2018

Review of Electrostatics (Cont'd)

Cylindrical Coordinates

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = P(\rho)\, Q(\phi)\, Z(z) \Rightarrow \frac{1}{P}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2 Q}{d\phi^2} +$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

$$\begin{cases} \frac{1}{Q}\frac{d^2 Q}{d\phi^2} = -\nu^2 \Rightarrow Q(\phi) = Ae^{i\nu\phi} + Be^{-i\nu\phi} \\ \frac{1}{Z}\frac{d^2 Z}{dz^2} = k^2 \Rightarrow Z(z) = Ce^{kz} + De^{-kz} \\ \rho^2 P''(\rho) + \rho P'(\rho) + (k^2\rho^2 - \nu^2)\, P(\rho) = 0 \end{cases}$$

Regarding $Q(\phi)$, if the entire range $0 \leq \phi < 2\pi$ is involved, then ν must be an integer.

Solutions to the radial equation (for real k) are Bessel functions of the first and second types, $J_\nu(k\rho)$ and $N_\nu(k\rho)$ respectively. $N_\nu(k\rho)$ is also called Neumann function.

The two linearly independent solutions are:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

The asymptotic behavior of Bessel functions is as follows:

$$x \ll 1 \Rightarrow \begin{cases} N_\nu(x) \to \frac{2}{\pi}\ln x \quad (\nu=0), \quad -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \quad (\nu \neq 0) \\ J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \end{cases}$$

$$x \gg 1 \Rightarrow \begin{cases} J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\ N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \end{cases}$$

Note that $N_\nu(x) \to \infty$ as $x \to 0$.

The pair J_ν, N_ν can be swapped for Bessel functions of the third kind, also called Hankel functions, according to:

$$H_\nu^{(1)} = J_\nu + iN_\nu \quad , \quad H_\nu^{(2)} = J_\nu - iN_\nu$$

$J_\nu(x)$ has an infinite number of roots:

$$J_\nu(x_{\nu n}) = 0 \qquad n = 1, 2, 3 \qquad x_{\nu n} \underset{(\text{when } \nu \geq 3)}{\simeq} n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}$$

Assuming that the set of Bessel functions is complete, we can expand an arbitrary function of ρ on the interval $0 \le \rho \le a$ in a Fourier-Bessel series:

$$f(\rho) = \sum_{n=1}^{\infty} C_{\nu n} J \left(\frac{x_{\nu n}}{a}\rho\right)$$

Where:

$$C_{\nu n} = \frac{2}{a^2 J^2_{\nu+1}(x_{\nu n})} \int_0^a f(\rho)\, J_\nu\left(\frac{x_{\nu n}}{a}\rho\right) \rho\, d\rho$$

Also, a Hankel transform pair $f(\rho)$, $F(k)$ are related to each other as follows:

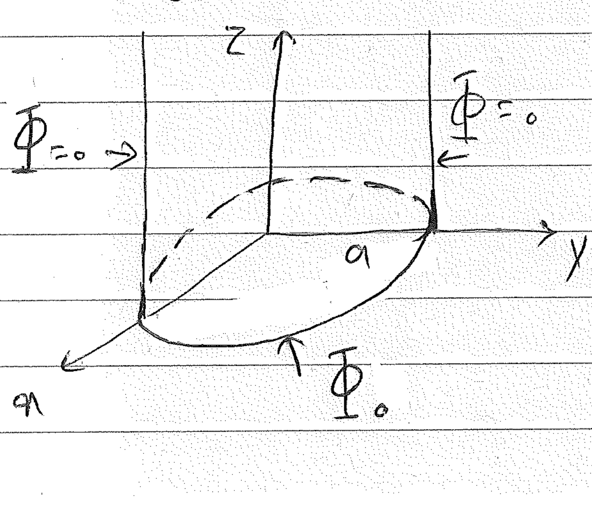
$$F_\nu(k) = \int_0^\infty f(\rho)\, J_\nu(k\rho)\, \rho\, d\rho \Rightarrow f(\rho) = \int_0^\infty F_\nu(k)\, J_\nu(k\rho)\, k\, dk$$

Example: The potential inside a semi-infinite cylinder whose cylindrical surface is grounded while the base has potential Φ_0.

$0 \le \rho \le a$

$0 \le \phi < 2\pi$

$0 \le z < \infty$



Since $0 \leq \phi < 2\pi$, then ν must be an integer m. Also, because $z \to \infty$, $Z \propto e^{-kz}$ where $k > 0$. Finally, since $\rho = 0$ is included, only $J_m(k\rho)$ with $m = 0, 1, 2, \ldots$ is permitted.

We can further use the azimuthal symmetry to conclude that only $m = 0$ can be present. Therefore:

$$\Phi(\rho, z) = \sum_{k>0} A_k J_0(k\rho) e^{-kz}$$

But $\Phi(a, z) = 0$, which implies that $J_0(ka) = 0$. Thus:

$$k_n = \frac{x_{0n}}{a} \qquad n = 1, 2, \ldots$$

$$\Phi(\rho, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{x_{0n}}{a}\rho\right) e^{-\frac{x_{0n}}{a} z}$$

Using the boundary condition at base, we have:

$$\Phi_0 = \sum_{n=1}^{\infty} A_n J_0\left(\frac{x_{0n}}{a}\rho\right) \Rightarrow A_n = \frac{2}{a^2 J_1^2(x_{0n})} \int_0 \Phi_0 J_0\left(\frac{x_{0n}}{a}\rho\right) \rho \, d\rho$$

Making a change of variable $u \equiv \frac{x_{0n}}{a}\rho$, results in the following integral:

$$\int_0^{\eta_{0n}} \underbrace{J_0(u)\, u\, du}_{\frac{d}{du}\left(J_1(u)\, u\right)} = \eta_{0n} J_1(\eta_{0n})$$

Hence:

$$A_n = \frac{2\Phi_0}{\eta_{0n}^{\cancel{2}}\, J_1^{\cancel{2}}(\eta_{0n})}\, \cancel{\eta_{0n}}\, \cancel{J_1(\eta_{0n})} = \frac{2\Phi_0}{\eta_{0n} J_1(\eta_{0n})}$$

The complete solution then is:

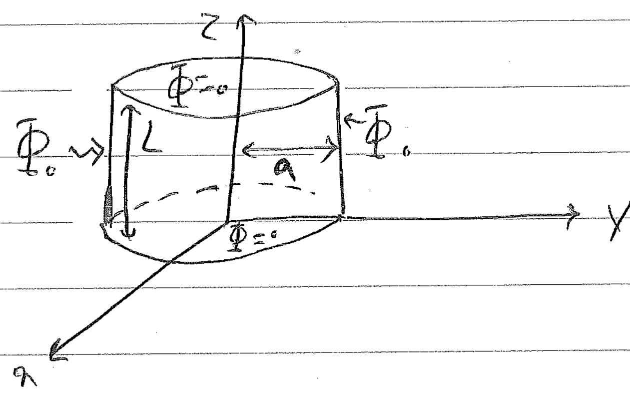
$$\boxed{\Phi(\rho, z) = 2\Phi_0 \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\eta_{0n}}{a}\rho\right)}{\eta_{0n} J_1(\eta_{0n})}\, e^{-\frac{\eta_{0n}}{a} z}}$$

Example: The potential inside a finite cylinder whose top and bottom faces are grounded, while the side has potential Φ_0.

$0 \le \rho \le a$

$0 \le \phi < 2\pi$

$0 \le z \le L$



We note that in this case the boundary condition at $z=0, L$ requires that $Z(z) \propto \sin kz, \cos kz$ (or $k^2 < 0$ in the expression on page 26).

For $k^2 < 0$, we have modified Bessel functions defined as follows:

$$I_m(x) = i^{-m} J_m(ix) \quad , \quad K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(x)$$

For small x ($x \ll 1$), the asymptotic behavior of the modified Bessel functions is as follows:

$$x \ll 1 \Longrightarrow \begin{cases} I_m(x) \to \dfrac{1}{\Gamma(m+1)} \left(\dfrac{x}{2}\right)^m \\ K_m(x) \to -\ln\left(\dfrac{x}{2}\right) \quad (m=0) \quad , \quad \dfrac{\Gamma(m)}{2} \left(\dfrac{2}{x}\right)^m \quad (m \neq 0) \end{cases}$$

In this example, $\rho = 0$ is included in the volume of interest, and hence only $I_m(k\rho)$ is permitted.

Due to the azimuthal symmetry, Φ does not depend on ϕ, which singles out the $m=0$ term. Also, since $\Phi(\rho, 0) = \Phi(\rho, L) = 0$, we must have:

$$Z(z) \propto \sin\left(\frac{n\pi}{L} z\right) \qquad n = 1, 2, \ldots$$

The most general solution then has the following form:

$$\Phi(\rho, z) = \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi}{L}\rho\right) \sin\left(\frac{n\pi}{L} z\right)$$

Imposing the boundary condition at $\rho = a$, we find:

$$\Phi_0 = \sum_{n=1}^{\infty} A_n \, I_0\left(\frac{n\pi}{L} a\right) \sin\left(\frac{n\pi}{L} z\right)$$

This results in:

$$A_n \, I_0\left(\frac{n\pi}{L} a\right) = \frac{2}{L} \int_0^L \Phi_0 \sin\left(\frac{n\pi}{L} z\right) dz = \frac{2\Phi_0}{\cancel{L}} \times \frac{\cancel{L}}{n\pi} \left[1 - (-1)^n\right]$$

$$\Rightarrow \boxed{A_n = \frac{2\Phi_0}{I_0\left(\frac{n\pi}{L} a\right) n\pi} \left[1 - (-1)^n\right]}$$